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SHALLOW WAVES IN A TWO-LAYER VORTEX FLUID UNDER A LID

A. A. Chesnokov

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A mathematical model of the vortex motion of an ideal two-layer fluid in a narrow straight channel is considered. The fluid motion in the Eulerian–Lagrangian coordinate system is described by quasilinear integrodifferential equations. Transformations of a set of the equations of motion which make it possible to apply the general method of studying integrodifferential equations of shallow-water theory, which is based on the generalization of the concepts of characteristics and the hyperbolicity for systems with operator functionals, are found. A characteristic equation is derived and analyzed. The necessary hyperbolicity conditions for a set of equations of motion of flows with a monotone-in-depth velocity profile are formulated. It is shown that the problem of sufficient hyperbolicity conditions is equivalent to the solution of a certain singular integral equation. In addition, the case of a strong jump in density (a heavy fluid in the lower layer and a quite lightweight fluid in the upper layer) is considered. A modeling that results in simplification of the system of equations of motion with its physical meaning preserved is carried out. For this system, the necessary and sufficient hyperbolicity conditions are given.

1. Derivation of a System of Equations of Motion. The solution of the initial boundary-value problem

$$\begin{gathered}
u_T + uu_X + vu_Y + \rho_1^{-1} p_X = 0, \quad \varepsilon^2(v_T + uv_X + vv_Y) + \rho_1^{-1} p_Y = -1, \quad 0 \leqslant Y \leqslant h, \\
u_T + uu_X + vu_Y + \rho_2^{-1} p_X = 0, \quad \varepsilon^2(v_T + uv_X + vv_Y) + \rho_2^{-1} p_Y = -1, \quad h \leqslant Y \leqslant 1, \\
u_X + v_Y = 0, \quad -\infty < X < \infty, \quad h_T + u^{\pm}(T, X, h) h_X = v^{\pm}(T, X, h), \\
v(T, X, 0) = v(T, X, 1) = 0, \quad u(0, X, Y) = u_0(X, Y), \quad h(0, X) = h_0(X)
\end{gathered} \tag{1.1}$$

describes the plane-parallel motion of an ideal two-layer fluid in a channel with walls $Y = 0$ and $Y = 1$ in the gravity field. The variables $u^1 = (gH_0)^{1/2} u$, $v^1 = (gH_0)^{1/2} H_0 L_0^{-1} v$, $\rho_i^1 = R_0 \rho_i$ $(i = 1, 2)$, $p^1 = R_0 g H_0 p$, $T^1 = L_0 (gH_0)^{-1/2} T$, $X^1 = L_0 X$, and $Y^1 = H_0 Y$ are the dimensional components of the velocity vector, the density, the pressure, the time, and the Cartesian coordinates, respectively, and u, v, ρ_i, p, T, X, and Y are the corresponding dimensionless quantities. The parameter L_0 determines the characteristic horizontal scale, H_0 is the depth of the channel, g is the acceleration of gravity, R_0 has the dimension of density, and $h(T, X)$ is the line of separation of the fluids with densities ρ_1 and ρ_2 $(0 \leqslant h \leqslant 1)$. The specified functions u_0 and h_0, where $0 \leqslant h_0 \leqslant 1$, are found for $0 \leqslant Y \leqslant 1$ and $-\infty < X < \infty$. The quantities $f^+(h)$ and $f^-(h)$ are the limits of the function $f(Y)$ as Y tends to h, so that $Y \geqslant h$ and $Y \leqslant h$, respectively.

In the narrow-channel approximation, the parameter $\varepsilon = H_0 L_0^{-1}$ is considered small, and terms of the order ε^2 are ignored in Eqs. (1.1). Owing to this, the pressure is distributed hydrostatically and can be represented in the form

$$\begin{gathered}
p(T, X, Y) = \rho_1(h - Y) + \rho_2(1 - h) + p^*(T, X) \qquad [0 \leqslant Y \leqslant h(T, X)], \\
p(T, X, Y) = \rho_2(1 - Y) + p^*(T, X) \qquad [h(T, X) \leqslant Y \leqslant 1],
\end{gathered} \tag{1.2}$$

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where $p^*(T,X)$ is the dimensionless pressure at the upper boundary of the channel. With allowance for the boundary conditions, integration of the continuity equation gives

$$v = -\int_0^Y u_X\,dY \quad (0 \leqslant Y \leqslant h), \qquad v = -\int_1^Y u_X\,dY \quad (h \leqslant Y \leqslant 1).$$

The jump in the function f upon passage through the line $Y = h(T,X)$ is denoted by $[f] = f^+(h) - f^-(h)$. It follows from Eqs. (1.1) that $[u_n] = 0$ [$u_n = -u(h)h_X + v(h)$ is the normal component of the velocity vector]. We consider flows in which $[u_n] = 0$, $[p] = 0$, and $[\rho]$ and, probably, $[u_\sigma]$ (the tangent component of the velocity vector), are not zero. Using the representation of the function v, one can write the condition $[u_n] = 0$ in the form

$$\Big(\int_0^h u\,dY + \int_h^1 u\,dY\Big)_X = 0.$$

After transformations, it is necessary to find u, p^*, and h:

$$\rho_1(u_T + uu_X + vu_Y) + (\rho_1 - \rho_2)h_X + p^*_X = 0 \qquad [0 \leqslant Y \leqslant h(T,X)],$$

$$\rho_2(u_T + uu_X + vu_Y) + p^*_X = 0 \quad (h(T,X) \leqslant Y \leqslant 1), \qquad h_T + \Big(\int_0^h u\,dY\Big)_X = 0, \tag{1.3}$$

$$\Big(\int_0^h u\,dY + \int_h^1 u\,dY\Big)_X = 0, \quad u(0,X,Y) = u_0(X,Y), \quad h(0,X) = h_0(X)$$

(the functions p and v are defined above). In this model, the absence of vorticity is equivalent to the condition $u_Y = 0$. We consider vortex flows with a monotone-in-depth velocity profile. For clarity, let $u_Y > 0$ in each layer.

We pass to the following Eulerian–Lagrangian coordinate system $x\ \lambda$ [1]:

$$T = t, \quad X = x, \quad Y = \Phi(t,x,\lambda) \qquad (0 \leqslant \lambda \leqslant 1). \tag{1.4}$$

Here $\Phi(t,x,\lambda)$ is the solution of the Cauchy problem,

$$\Phi_t + u(t,x,\Phi)\Phi_x = v(t,x,\Phi), \qquad \Phi(0,x,\lambda) = \Phi_0(x,\lambda). \tag{1.5}$$

For $0 \leqslant \lambda \leqslant 1/2$ and $1/2 \leqslant \lambda \leqslant 1$, the function Φ_0 is defined by the formulas $\Phi_0(x,\lambda) = 2\lambda h_0(x)$, and $\Phi_0(x,\lambda) = 2(1-\lambda)h_0(x) + 2(\lambda - 1/2)$. The line $Y = h(T,X)$ becomes a straight line $\lambda = 1/2$, and, for $\lambda = 0$, $1/2$, and 1, the function $\Phi(t,x,\lambda)$ takes on the values 0, $h(T,X)$, and 1, respectively. By virtue of (1.2)–(1.4), to determine the functions $u(t,x,\lambda)$, $h(t,x)$, and $H(t,x,\lambda) = \Phi_\lambda$ we obtain the system

$$\rho_1(u_t + uu_x) + (\rho_1 - \rho_2)h_x - \rho_2(u_{1t} + u_1u_{1x}) = 0 \qquad (0 \leqslant \lambda \leqslant 1/2),$$

$$u_t + uu_x - (u_{1t} + u_1u_{1x}) = 0 \quad (1/2 \leqslant \lambda \leqslant 1), \qquad H_t + (uH)_x = 0, \quad \int_0^{1/2} H\,d\lambda = h(t,x), \tag{1.6}$$

$$\int_{1/2}^1 H\,d\lambda = 1 - h(t,x), \qquad u(0,x,\lambda) = u_0(x,\Phi_0(x,\lambda)), \qquad H(0,x,\lambda) = \Phi_{0\lambda}(x,\lambda).$$

Here $u_1 = u(t,x,1)$ and $p^*_x = -\rho_2(u_{1t} + u_1u_{1x})$; since the pressure p^*_x does not depend on λ, it can be expressed in terms of the velocity u and its derivatives for fixed λ. The change in the variables (1.4) is reversible under the condition that $\Phi_\lambda \neq 0$. We assume that $\Phi_\lambda > 0$.

We shall consider flows in which $u^+(t,x,1/2) > u^-(t,x,1/2)$. For further transformation of the equations of motion, we shall pass from the functions $u(t,x,\lambda)$, $H(t,x,\lambda)$, and $h(t,x)$ to new desired quantities

$$u_\lambda,\ \theta = Hu_\lambda^{-1}, \qquad w(t,x) = \rho_1 u^-(t,x,1/2) - \rho_2 u^+(t,x,1/2)$$

(the variable θ is inversely proportional to the vorticity with the minus sign) with the use of the relation

$$\int_0^{1/2} uH\,d\lambda + \int_{1/2}^{1} uH\,d\lambda = Q(t). \tag{1.7}$$

Let the function $Q(t)$ be defined. One can pass from Eqs. (1.6) to the system

$$\begin{gathered} u_{\lambda t} + uu_{\lambda x} + u_\lambda u_x = 0, \qquad \theta_t + u\theta_x = 0 \qquad (0 \leqslant \lambda \leqslant 1), \\ w_t + \rho_1^{-1}(w+\rho_2 q)w_x + \rho_1^{-1}\rho_2(w-(\rho_1-\rho_2)q)q_x + (\rho_1-\rho_2)\int_0^{1/2}(u_{\lambda x}\theta + u_\lambda\theta_x)\,d\lambda = 0, \end{gathered} \tag{1.8}$$

where, according to (1.7), the functions $u(t,x,\lambda)$, $q(t,x) = u^+(t,x,1/2)$, u_x, and q_x, are expressed by the formulas

$$\begin{gathered} u = \rho_1^{-1}(w+\rho_2 q) - \int_\lambda^{1/2} u_\nu\,d\nu, \quad u_x = \rho_1^{-1}(w_x+\rho_2 q_x) - \int_\lambda^{1/2} u_{\nu x}\,d\nu \qquad (0 \leqslant \lambda \leqslant 1/2), \\ u = q + \int_{1/2}^{\lambda} u_\nu\,d\nu, \quad u_x = q_x + \int_{1/2}^{\lambda} u_{\nu x}\,d\nu \qquad (1/2 \leqslant \lambda \leqslant 1), \\ q = \Big(\rho_1^{-1}\rho_2\int_0^{1/2} u_\lambda\theta\,d\lambda + \int_{1/2}^{1} u_\lambda\theta\,d\lambda\Big)^{-1}\Big[Q(t) + \int_0^{1/2} u_\lambda\theta\Big(\int_\lambda^{1/2} u_\nu\,d\nu\Big)\,d\lambda \\ -\rho_1^{-1}w\int_0^{1/2} u_\lambda\theta\,d\lambda - \int_{1/2}^{1} u_\lambda\Big(\int_\lambda^{1} u_\nu\theta\,d\nu\Big)\,d\lambda\Big], \end{gathered} \tag{1.9}$$

$$\begin{gathered} q_x = \Big(\rho_1^{-1}\rho_2\int_0^{1/2} u_\lambda\theta\,d\lambda + \int_{1/2}^{1} u_\lambda\theta\,d\lambda\Big)^{-1}\Big[-q\Big(\rho_1^{-1}\rho_2\int_0^{1/2}(u_\lambda\theta)_x\,d\lambda + \int_{1/2}^{1}(u_\lambda\theta)_x\,d\lambda\Big) \\ + \int_0^{1/2}\Big(u_\lambda\theta\Big(\int_\lambda^{1/2} u_\nu\,d\nu\Big)\Big)_x d\lambda - \rho_1^{-1}\Big(w\int_0^{1/2} u_\lambda\theta\,d\lambda\Big)_x - \int_{1/2}^{1}\Big(u_\lambda\Big(\int_\lambda^{1} u_\nu\theta\,d\nu\Big)\Big)_x d\lambda\Big]. \end{gathered}$$

The initial conditions for system (1.8) have the form

$$u_\lambda(0,x,\lambda) = u_{0\lambda}, \quad \theta(0,x,\lambda) = \Phi_{0\lambda}/u_{0\lambda}, \quad w(0,x) = \rho_1 u_0^+(x,h_0(x)) - \rho_2 u_0^-(x,h_0(x)).$$

If the functions u_λ, θ, and w are found, we know $H = u_\lambda\theta$, the line of separation of the fluids $h(t,x) = \int_0^{1/2} u_\lambda\theta\,d\lambda$, and u by virtue of (1.9). The integration of the quantity $u_\lambda\theta$ from 0 to 1 gives the total depth (upper boundary) of the channel. The initial data are such that, for $t = 0$, the depth of the channel is equal to 1. This value is preserved at all moments of time, because

$$\Big(\int_0^{1/2} u_\lambda\theta\,d\lambda + \int_{1/2}^{1} u_\lambda\theta\,d\lambda\Big)_t = -\Big(\int_0^{1/2} uu_\lambda\theta\,d\lambda + \int_{1/2}^{1} uu_\lambda\theta\,d\lambda\Big)_x = -(Q(t))_x = 0.$$

The formulas $p_x^* = -\rho_2(u_{1t} + u_1 u_{1x})$, (1.2), $\Phi_\lambda = H$, (1.4), and (1.5) allow one to find the pressure (to an arbitrary function of t), Φ, and the vertical component of the velocity vector v. System (1.1) is reduced to problem (1.8) by transformations, and its hyperbolicity properties can be studied by the methods of [2].

2. Characteristic Properties of Eqs. (1.8). A characteristic equation is derived and the necessary hyperbolicity conditions are formulated. The problem of obtaining the sufficient conditions is reduced to the solution of a singular integral equation.

System (1.8) can be written in the form

$$\boldsymbol{U}_t + A\boldsymbol{U}_x = 0, \tag{2.1}$$

where $\boldsymbol{U} = (u_\lambda, \theta, w)^t$, A is the matrix with operator functionals, which arises when (1.9) are substituted into (1.8) and act on the vector function $\boldsymbol{f}$ according to the rule

$$A\boldsymbol{f} = \Big(uf_1 - u_\lambda \int\limits_\lambda^{1/2} f_1\, d\nu + u_\lambda(\rho_1^{-1} f_3 + \rho_1^{-1}\rho_2 N)s(1/2-\lambda)$$
$$+u_\lambda N s(\lambda - 1/2), uf_2, \rho_1^{-1}(w+\rho_2 q) f_3$$
$$+\rho_1^{-1}\rho_2(w - (\rho_1-\rho_2)q)N + (\rho_1-\rho_2)\int\limits_0^{1/2}(f_1\theta + u_\lambda f_2)\, d\lambda\Big)^t.$$

In this expression, $s(\lambda)$ is a stepwise function equal to 0 for $\lambda < 0$ and 1 for $\lambda > 0$, and N has the form

$$N = ((\rho_1^{-1}\rho_2 - 1)h + 1)^{-1}\Big[\int\limits_0^{1/2}\theta\Big(u\int\limits_\lambda^{1/2} f_1\, d\nu\Big)_\lambda d\lambda - \int\limits_{1/2}^{1} f_1\Big(\int\limits_\lambda^{1} u_\nu\theta\, d\nu\Big)\, d\lambda$$
$$-\int\limits_{1/2}^{1} u\theta f_1\, d\lambda - \int\limits_0^{1/2} uu_\lambda f_2\, d\lambda - \int\limits_{1/2}^{1} uu_\lambda f_2\, d\lambda - \rho_1^{-1} h f_3\Big].$$

According to [2], the characteristic of system (2.1) is determined by the differential equation $x'(t) = k(t,x)$, where k is the eigenvalue of the operator A^*). The solution of the equation

$$(\boldsymbol{F}, (A - kI)\varphi) = 0 \tag{2.2}$$

relative to the vector functional $\boldsymbol{F} = (F_1^\mp, F_2^\mp, F_3)$ which acts over the variable λ on an arbitrary, indefinitely differentiable vector function $\varphi = (\varphi_1, \varphi_2, \varphi_3)^t$ for $0 \leqslant \lambda \leqslant 1/2$ and $\varphi = (\psi_1, \psi_2, \varphi_3)^t$ for $1/2 \leqslant \lambda \leqslant 1$ [the functions φ_i and ψ_i ($i = 1, 2$) for $\lambda = 1/2$ have different values, φ_3 does not depend on λ, and the dependence on t and x as the parameters) is sough in the class of generalized or locally integrable functions. The expression $(\boldsymbol{F}, \varphi) = (F_1^-, \varphi_1) + (F_2^-, \varphi_2) + (F_1^+, \psi_1) + (F_2^+, \psi_2) + (F_3, \varphi_3)$ denotes the result of the action of the functional $\boldsymbol{F}$ on the trial vector function. We consider the functions u_λ and θ indefinitely differentiable with respect to λ in each layer. The action of the functional $\boldsymbol{F}$ on Eq. (2.1) gives a relation on the characteristic

$$(\boldsymbol{F}, \boldsymbol{U}_t + k\boldsymbol{U}_x) = 0. \tag{2.3}$$

The system is hyperbolic if all eigenvalues k are real and the set of relations on the characteristics (2.3) is equivalent to Eqs. (2.1).

With allowance for the independence of the components of the trial vector function on Eqs. (2.2), we obtain the equalities

$$\Big(F_1^-, (u-k)\varphi_1 - u_\lambda\int\limits_\lambda^{1/2}\varphi_1\, d\nu\Big) + M\int\limits_0^{1/2}\theta\Big(u\int\limits_\lambda^{1/2}\varphi_1\, d\nu\Big)_\lambda d\lambda + (\rho_1-\rho_2)(F_3, 1)\int\limits_0^{1/2}\theta\varphi_1\, d\lambda = 0; \tag{2.4}$$

$$\left(F_1^+, (u-k)\psi_1 + u_\lambda \int\limits_{1/2}^{\lambda} \psi_1\, d\nu\right) - M\left[\int\limits_{1/2}^{1} \psi_1\left(\int\limits_{\lambda}^{1} u_\nu \theta\, d\nu\right) d\lambda + \int\limits_{1/2}^{1} u\theta\psi_1\, d\lambda\right] = 0; \tag{2.5}$$

$$(F_2^-, (u-k)\varphi_2) - M\int\limits_0^{1/2} u u_\lambda \varphi_2\, d\lambda + (\rho_1 - \rho_2)(F_3, 1)\int\limits_0^{1/2} u_\lambda \varphi_2\, d\lambda = 0; \tag{2.6}$$

$$(F_2^+, (u-k)\psi_2) - M\int\limits_{1/2}^{1} u u_\lambda \psi_2\, d\lambda = 0; \tag{2.7}$$

$$-hM\varphi_3 + [(F_1^-, u_\lambda) + (w + \rho_2 q - \rho_1 k)(F_3, 1)]\varphi_3 = 0, \tag{2.8}$$

in which $M = ((\rho_1^{-1}\rho_2 - 1)h + 1)^{-1}[\rho_1^{-1}\rho_2(F_1^-, u_\lambda) + (F_1^+, u_\lambda) + \rho_1^{-1}\rho_2(w - \rho_1 q + \rho_2 q)(F_3, 1)]$.

We consider a set of numbers k which belong to a complex plane, except for the segments $[u_0, u_{1/2}^-]$ and $[u_{1/2}^+, u_1]$. We express the quantity $(F_3, 1)$ from Eq. (2.8):

$$(F_3, 1) = (h(F_1^+, u_\lambda) - (1-h)(F_1^-, u_\lambda))K, \tag{2.8'}$$

where $K = [(1-h)(w + \rho_2 q - \rho_1 k) + \rho_2 h(q-k)]^{-1}$. Using (2.8′) and the functions

$$\varphi = \left[(u-k)\int\limits_{\lambda}^{1/2} \varphi_1\, d\nu\right]_\lambda, \qquad \psi = \left[(u-k)\int\limits_{1/2}^{\lambda} \psi_1\, d\nu\right]_\lambda$$

we reduce expressions (2.4) and (2.5) to the form

$$(F_1^-, \varphi) = -K\Big[(\rho_2(q-k)(F_1^-, u_\lambda) + (w + \rho_2 q - \rho_1 k)(F_1^+, u_\lambda))\int\limits_0^{1/2} \theta\Big(u(u-k)^{-1}\int\limits_{\lambda}^{1/2} \varphi\, d\nu\Big)_\lambda d\lambda$$

$$+(\rho_1 - \rho_2)((1-h)(F_1^-, u_\lambda) - h(F_1^+, u_\lambda))\int\limits_0^{1/2} \theta\Big((u-k)^{-1}\int\limits_{\lambda}^{1/2} \varphi\, d\nu\Big)_\lambda d\lambda\Big]; \tag{2.4'}$$

$$(F_1^+, \psi) = K\Big[(\rho_2(q-k)(F_1^-, u_\lambda) + (w + \rho_2 q - \rho_1 k)(F_1^+, u_\lambda))\int\limits_{1/2}^{1} \theta\Big(u(u-k)^{-1}\int\limits_{1/2}^{\lambda} \psi\, d\nu\Big)_\lambda d\lambda\Big]. \tag{2.5'}$$

Assuming $\varphi = u_\lambda$ and $\psi = u_\lambda$ in formulas (2.4′) and (2.5′) and equating to zero the determinant of the linear system which is uniform relative to (F_1^-, u_λ) and (F_1^+, u_λ), we obtain the characteristic equation

$$k\Big[\rho_2\int\limits_0^{1/2} (u-k)^{-2} u_\lambda \theta\, d\lambda + \rho_1\int\limits_{1/2}^{1} (u-k)^{-2} u_\lambda \theta\, d\lambda$$

$$-(\rho_1 - \rho_2)\int\limits_0^{1/2} (u-k)^{-2} u_\lambda \theta\, d\lambda \int\limits_{1/2}^{1} (u-k)^{-2} u_\lambda \theta\, d\lambda\Big] = 0 \tag{2.9}$$

which determines the discrete spectrum of the operator A^*. The expression in square brackets in formula (2.9) is denoted by $\chi(k)$.

Remark 1. The characteristic equation (2.9) (without the cofactor k) can be derived using Eqs. (1.6) and the fact that, on the characteristics, the derivatives of the desired functions along the normal $\partial_n = k(t,x)\partial_t - \partial_x$ are not expressed in terms of the derivatives along the tangent $\partial_\sigma = \partial_t + k(t,x)\partial_x$.

For $\rho_1 < \rho_2$, the characteristic equation (2.9), $k\chi(k) = 0$, has one real root $k = 0$, because $u_\lambda\theta = H = \Phi_\lambda > 0$ and the function $\chi(k)$ contains only positive terms ($k \neq \pm\infty$). We shall show that, in the case $\rho_1 > \rho_2$

(a heavier fluid in the lower layer and a lighter-weight fluid in the upper layer), the equation $\chi(k) = 0$ on the solution considered can have real roots $k = k^i$. It is convenient to analyze the characteristic equation with the use of the functions

$$a(k) = \rho_2 - (\rho_1 - \rho_2) \int_{1/2}^{1} (u - k)^{-2} u_\lambda \theta \, d\lambda, \qquad b(k) = \rho_1 - (\rho_1 - \rho_2) \int_{0}^{1/2} (u - k)^{-2} u_\lambda \theta \, d\lambda$$

in terms of which $\chi(k) = (\rho_1 - \rho_2)^{-1}(\rho_1\rho_2 - a(k)b(k))$. The function $a(k)$ does not exceed ρ_2 and tends to this value when k tends in modulus to infinity. By virtue of the sign of the derivative, $a(k)$ decreases monotonically on the interval $(-\infty, u^+_{1/2})$ and increases monotonically on the interval (u_1, ∞). The behavior of the function $b(k)$ on the intervals $(-\infty, u_0)$ and $(u^-_{1/2}, \infty)$ is similar $[b(k) \leqslant \rho_1]$. For quite large, in modulus, values of k, the function $\chi(k)$ is greater than zero, because $0 < a(k) < \rho_2$ and $0 < b(k) < \rho_1$. If $a(u_0) < 0$, $\chi(k) \to -\infty$ as $k \to u_0$, and, hence, the function $\chi(k)$ on the interval $(-\infty, u_0)$ has at least one root. Since the derivative of the function $a(k)b(k)$ on this interval changes sign only once (between the zeros of the functions a and b), the equation $\chi(k) = 0$ has a unique root for $k < u_0$. If $a(u_0) > 0$, then, for $k < u_0$, the function χ does not vanish. Similarly, one can show whether the function χ vanishes or not for $k > u_1$, depending on the sign of $b(u_1)$. In addition, the function χ can vanish in the gap $(u^-_{1/2}, u^+_{1/2})$.

In the vortex-free case, for $\varepsilon = 0$, Eqs. (1.1) are reduced to a system of two differential equations [3] which have two real roots in the hyperbolicity domain. Therefore, in this case, the situation in which there are two real roots $k^1 < u_0$ and $k^2 > u_1$ on the solution considered is, probably, the most natural. The conditions

$$1)\ \ a(u_0) < 0, \quad b(u_1) < 0; \qquad 2)\ \ a(u^-_{1/2}) < -(\rho_1\rho_2)^{1/2}, \quad b(u^+_{1/2}) < -(\rho_1\rho_2)^{1/2}$$

are sufficient for the existence of the roots $k^1 < u_0$ and $k^2 > u_1$. Conditions No. 1 guarantee the existence of these and only these roots on the intervals $(-\infty, u_0)$ and (u_1, ∞), while conditions No. 2 (if conditions No. 1 are satisfied) do not allow the function χ to vanish in the gap $[u^-_{1/2}, u^+_{1/2}]$.

We calculate the eigenfunctionals $\boldsymbol{F}^i$ and $\boldsymbol{F}^0$ which correspond to the eigenvalues $k = k^i$ $(k^i \neq 0)$ and $k = 0$. With account of (2.9), from Eqs. (2.4′) and (2.5′), we find the action of the first component of the functional $\boldsymbol{F}^i$ on the trial function:

$$(F_1^{i-}, \varphi) = \int_0^{1/2} \theta\varphi \, d\lambda - k^i \rho_2^{-1} a(k^i) \int_0^{1/2} \theta \Big[(u - k^i)^{-1} \int_\lambda^{1/2} \varphi \, d\nu \Big]_\lambda d\lambda,$$

$$(F_1^{i+}, \psi) = \int_{1/2}^{1} \theta \Big[u(u - k^i)^{-1} \int_{1/2}^{\lambda} \psi \, d\nu \Big]_\lambda d\lambda.$$

Returning to (2.6), (2.7), and (2.8′), we find $F_2^{i\mp}$ and F_3^i:

$$(F_2^{i-}, \varphi) = \int_0^{1/2} u_\lambda \varphi \, d\lambda + k^i \rho_2^{-1} a(k^i) \int_0^{1/2} (u - k^i)^{-1} u_\lambda \varphi \, d\lambda,$$

$$(F_2^{i+}, \psi) = \int_{1/2}^{1} (u - k^i)^{-1} u u_\lambda \psi \, d\lambda, \quad (F_3^i, 1) = k^i \rho_2^{-1} \int_{1/2}^{1} (u - k^i)^{-2} u_\lambda \theta \, d\lambda.$$

Assuming that $k^i = 0$ in these formulas, we determine the functional $\boldsymbol{F}^0$:

$$(F_1^{0-}, \varphi) = \int_0^{1/2} \theta\varphi \, d\lambda, \qquad (F_1^{0+}, \psi) = \int_{1/2}^{1} \theta\psi \, d\lambda,$$

$$(F_2^{0-},\varphi)=\int_0^{1/2} u_\lambda \varphi\, d\lambda, \qquad (F_2^{0+},\psi)=\int_{1/2}^{1} u_\lambda \psi\, d\lambda, \qquad (F_3^0,1)=0.$$

In addition to the discrete spectrum, the operator A^* has a continuous characteristic spectrum $k^\lambda = u(t,x,\lambda)$, which consists of two segments $[u_0, u_{1/2}^-]$ and $[u_{1/2}^+, u_1]$. By analogy, we use Eq. (2.8′) and the functions

$$\varphi=\left[(u'-u)\int_\nu^\lambda \varphi_1(\mu)\,d\mu\right]_\nu, \qquad \psi=\left[(u'-u)\int_\lambda^\nu \psi_1(\mu)\,d\mu\right]_\nu$$

to transform expressions (2.4) and (2.5), in which the functionals now act on the variable ν and $u=u(t,x,\nu)$ and $k=u(t,x,\lambda)$. Hereinafter, for brevity, $f=f(t,x,\lambda)$ and $f'=f(t,x,\nu)$. In this case, we obtain the eigenfunctional $\boldsymbol{F}^{1\lambda}$

$$(F_1^{1\lambda-},\varphi)=\int_0^{1/2}\theta'\varphi'\,d\nu-u\rho_1\rho_2^{-1}\int_\lambda^{1/2}\varphi(\mu)\,d\mu\int_{1/2}^{1}(u'-u)^{-2}u'_\nu\theta'\,d\nu$$

$$+u\rho_2^{-1}a(u)\int_0^{1/2}\theta'\left[(u'-u)^{-1}\int_\lambda^\nu\varphi(\mu)\,d\mu\right]_\nu d\nu,$$

$$(F_1^{1\lambda+},\psi)=\int_{1/2}^{1}\theta'\left[u'(u'-u)^{-1}\int_{1/2}^{\nu}\psi(\mu)\,d\mu\right]_\nu d\nu,$$

$$(F_2^{1\lambda-},\varphi)=\int_0^{1/2}u'_\nu\varphi'\,d\nu+u\rho_2^{-1}a(u)\int_0^{1/2}(u'-u)^{-1}u'_\nu\varphi'\,d\nu,$$

$$(F_2^{1\lambda+},\psi)=\int_{1/2}^{1}(u'-u)^{-1}u'u'_\nu\psi'\,d\nu, \qquad (F_3^{1\lambda},1)=u\rho_2^{-1}\int_{1/2}^{1}(u'-u)^{-2}u'_\nu\theta'\,d\nu$$

for $0\leqslant\lambda\leqslant 1/2$ and the eigenfunctional $\boldsymbol{F}^{2\lambda}$

$$(F_1^{2\lambda-},\varphi)=\int_0^{1/2}\theta'\varphi'\,d\nu-u\rho_1 b^{-1}(u)\int_0^{1/2}\theta'\left[(u'-u)^{-1}\int_\nu^{1/2}\varphi(\mu)\,d\mu\right]_\nu d\nu,$$

$$(F_1^{2\lambda+},\psi)=\int_{1/2}^{1}\theta'\left[u'(u'-u)^{-1}\int_\lambda^{\nu}\psi(\mu)\,d\mu\right]_\nu d\nu+u\rho_2 b^{-1}(u)\int_0^{1/2}(u'-u)^{-2}u'_\nu\theta'\,d\nu\int_{1/2}^{\lambda}\psi(\mu)\,d\mu,$$

$$(F_2^{2\lambda-},\varphi)=\int_0^{1/2}u'_\nu\varphi'\,d\nu+u\rho_1 b^{-1}(u)\int_0^{1/2}(u'-u)^{-1}u'_\nu\varphi'\,d\nu,$$

$$(F_2^{2\lambda+},\psi)=\int_{1/2}^{1}(u'-u)^{-1}u'u'_\nu\psi'\,d\nu, \qquad (F_3^{2\lambda},1)=-ub^{-1}(u)\int_0^{1/2}(u'-u)^{-2}u'_\nu\theta'\,d\nu$$

for $1/2\leqslant\lambda\leqslant 1$ (the functions a and b were defined above).

Obviously, Eqs. (2.4)–(2.8) have one more nontrivial solution

$$F_2^{3\lambda-}=\delta(\nu-\lambda), \qquad F_2^{4\lambda+}=\delta(\nu-\lambda),$$

and the other components of the vector functionals $\boldsymbol{F}^{3\lambda}$ and $\boldsymbol{F}^{4\lambda}$ are equal to zero.

To derive relations on the characteristics for $u \neq 0$, we act by the functionals $\boldsymbol{F}^0$, $\boldsymbol{F}^i$ and $\boldsymbol{F}^{j\lambda}$, where $j = 1$, 2, 3, and 4, on system (1.8). The action of the functional $\boldsymbol{F}^0$ gives the equation

$$\Big(\int_0^{1/2} u_\lambda \theta \, d\lambda + \int_{1/2}^{1} u_\lambda \theta \, d\lambda\Big)_t = 0 \tag{2.10}$$

which expresses the fact that the upper boundary of the channel is fixed (equal to 1 at the initial moment of time). With account of (2.10), the use of the functions

$$r^i = \int_0^{1/2} (u - k^i)^{-1} u_\lambda \theta \, d\lambda, \qquad l^i = \int_{1/2}^{1} (u - k^i)^{-1} u_\lambda \theta \, d\lambda,$$

$$R = \int_0^{1/2} (u' - u)^{-1} u'_\nu \theta' \, d\nu, \qquad L = \int_{1/2}^{1} (u' - u)^{-1} u'_\nu \theta' \, d\nu,$$

and after transformations, the result of the action of the functionals $\boldsymbol{F}^i$, $\boldsymbol{F}^{1\lambda}$, $\boldsymbol{F}^{2\lambda}$, $\boldsymbol{F}^{3\lambda}$, and $\boldsymbol{F}^{4\lambda}$ on system (1.8) takes the form

$$\begin{gathered} a(k^i)(r^i_t + k^i r^i_x) + \rho_2(l^i_t + k^i l^i_x) + (\rho_2 - a(k^i))(k^i + k^i k^i_x) = 0, \\ a(u)(R_t + uR_x) + \rho_2(L_t + uL_x) + (\rho_2 - a(u))(u + uu_x) = 0, \\ \rho_1(R_t + uR_x) + b(u)(L_t + uL_x) - (\rho_1 - b(u))(u + uu_x) = 0, \quad \theta_t + u\theta_x = 0. \end{gathered} \tag{2.11}$$

Equations (2.10) and (2.11) form a system of relations on the characteristics [the characteristic form (1.8)].

Remark 2. If u vanishes on the interval $[u_0, u^-_{1/2}]$, to derive relations on the characteristics, it is necessary to use the eigenfunctionals $\boldsymbol{F}^0$, $\boldsymbol{F}^i$, and $\boldsymbol{F}^{j\lambda}$, where $j = 2$, 3, and 4, and the eigenfunctionals $\boldsymbol{P}^{1\lambda}$ which possess the property $(\boldsymbol{P}^{1\lambda}, (A - uI)\varphi) = (\boldsymbol{F}^0, \varphi)$. Acting by these functionals on system (1.8), we obtain Eqs. (2.10) and (2.11). In the case where u vanishes on the interval $[u^+_{1/2}, u_1]$, we act similarly.

The necessary hyperbolicity conditions consist of the absence of complex characteristic roots on the solution considered. If the number of real solutions of the equation $\chi(k) = 0$ is known, the condition that the characteristic equation has no complex roots is formulated in terms of the complex function $\chi(z)$, to be exact, its limiting values

$$\chi^\pm(u) = a(u)\Big[-(u^-_{1/2} - u)^{-1}\theta^-_{1/2} + (u_0 - u)^{-1}\theta_0 + \int_0^{1/2} (u' - u)^{-1} \theta'_\nu \, d\nu\Big]$$

$$+\rho_1 \int_{1/2}^{1} (u' - u)^{-2} u'_\nu \theta' \, d\nu \pm \pi i a(u)\theta_\lambda / u_\lambda,$$

$$\chi^\pm(u) = b(u)\Big[-(u_1 - u)^{-1}\theta_1 + (u^+_{1/2} - u)^{-1}\theta^+_{1/2} + \int_{1/2}^{1} (u' - u)^{-1} \theta'_\nu \, d\nu\Big]$$

$$+\rho_2 \int_0^{1/2} (u' - u)^{-2} u'_\nu \theta' \, d\nu \pm \pi i b(u)\theta_\lambda / u_\lambda$$

from the upper and lower half-planes on the segments $[u_0, u^-_{1/2}]$ and $[u^+_{1/2}, u_1]$, respectively.

Lemma. *Equation* (2.9) *on the solution* u_λ, θ, w *has no complex roots if the condition*

$$\Delta \arg(\chi^+/\chi^-) = 2\pi(n - 2) \qquad (\chi^\pm \neq 0) \tag{2.12}$$

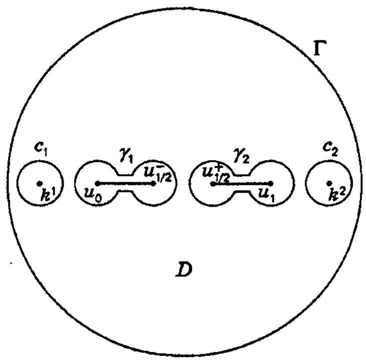

Fig. 1

is satisfied [the increment of the argument when λ *changes from* 0 *to* 1/2 *and from* 1/2 *to 1; n is the number of real zeros of the function* $\chi(k)$].

Proof. We enclose the intervals of variation of the function u in contours of the type of a dumb-bell γ_1 and γ_2, and the points k^i (the zeros of the function χ) in circles of small radius c_i and outline a circle Γ of such a large radius that all solutions of the equation $\chi(k) = 0$ lie in an appropriate circle (in the case of two real roots) (see Fig. 1). In the domain D (the intersection of the circle and the outside γ_1, γ_2, c_i), the function $\chi(k)$ is analytic and has no poles. By virtue of the principle of the argument, the number of zeros of $\chi(k)$ in this domain equals the increment of the argument over the contours divided by 2π. The increment of the argument counterclockwise along Γ is equal to -4π (second-order zeros). When we go around each circle of small radius, which bounds the points u_0, $u_{1/2}^-$, $u_{1/2}^+$, and u_1 clockwise, the argument is incremented by 2π (first-order poles) and -2π in the same direction of the circles c_i (first-order zeros). After summation, the increment of the argument along all the contours, except for the handles of the dumb-bells, equals $2\pi(2-n)$. Hence, to obtain a zero sum, the argument on the handles of the dumb-bells should be incremented by $2\pi(n-2)$. Thus, condition (2.12) is necessary and sufficient for the lack of complex solutions of the equation $\chi(k) = 0$ if this equation has n real roots. The lemma is proved.

To prove the equivalence of Eqs. (1.8) and the relations on the characteristics (2.10) and (2.11), it is necessary to show that the equalities $(\boldsymbol{F}^{j\lambda}, \boldsymbol{S}) = 0$ ($j = 1, 2, 3, 4$), $(\boldsymbol{F}^0, \boldsymbol{S}) = 0$, and $(\boldsymbol{F}^i, \boldsymbol{S}) = 0$ are satisfied if and only if the vector function $\boldsymbol{S} = (S_1, S_2, S_3)$ (the first two components are the functions of λ, and the last is a constant) is identically equal to zero.

It follows from equations $(\boldsymbol{F}^{3\lambda}, \boldsymbol{S}) = 0$ and $(\boldsymbol{F}^{4\lambda}, \boldsymbol{S}) = 0$ that the component S_2 is zero. In view of this, we write the results of the action of the functionals $\boldsymbol{F}^{1\lambda}$, $\boldsymbol{F}^{2\lambda}$, $\boldsymbol{F}^i$, and $\boldsymbol{F}^0$ on the vector function $\boldsymbol{S}$:

$$a(u)\int_0^{1/2} \theta'\Big[(u'-u)^{-1}\int_\lambda^\nu S_1\,d\mu\Big]_\nu d\nu - \rho_1\int_\lambda^{1/2} S_1\,d\mu\int_{1/2}^1 (u'-u)^{-2}u'_\nu\theta'\,d\nu$$
$$+\rho_2\int_{1/2}^1 \theta'\Big[(u'-u)^{-1}\int_{1/2}^\nu S_1\,d\mu\Big]_\nu d\nu + S_3\int_{1/2}^1 (u'-u)^{-2}u'_\nu\theta'\,d\nu = 0; \tag{2.13}$$

$$b(u)\int_{1/2}^1 \theta'\Big[(u'-u)^{-1}\int_\lambda^\nu S_1\,d\mu\Big]_\nu d\nu + \rho_2\int_{1/2}^\lambda S_1\,d\mu\int_0^{1/2} (u'-u)^{-2}u'_\nu\theta'\,d\nu$$
$$-\rho_1\int_0^{1/2} \theta'\Big[(u'-u)^{-1}\int_\nu^{1/2} S_1\,d\mu\Big]_\nu d\nu - S_3\int_0^{1/2} (u'-u)^{-2}u'_\nu\theta'\,d\nu = 0; \tag{2.14}$$

$$a(k^i) \int_0^{1/2} \theta\Big[(u-k^i)^{-1} \int_\lambda^{1/2} S_1\,d\nu\Big]_\lambda d\lambda - \rho_2 \int_{1/2}^{1} \theta\Big[(u-k^i)^{-1} \int_{1/2}^{\lambda} S_1\,d\nu\Big]_\lambda d\lambda$$

$$-S_3 \int_{1/2}^{1} (u-k^i)^{-2} u_\lambda \theta\,d\lambda = 0, \qquad \int_0^{1/2} \theta S_1\,\lambda + \int_{1/2}^{1} \theta S_1\,\lambda = 0. \tag{2.15}$$

By changing the variables $\tau(\lambda) = \int_\lambda^{1/2} S_1\,d\nu$ and by integrating by parts, one can reduce Eqs. (2.13) and (2.14) to a singular integral equation [4] specified on the disclosed contours and containing a characteristic part and a Fredholm operator of the first kind. Thus, S_3 and the integration constants τ_0 and τ_1 are determined from Eqs. (2.15) (we assume that there are at least two roots k^i). In [5, 6], the characteristic singular integral equations appeared in similar situations. They were solved by reducing them to a conjugation problem. In our case, it is impossible to solve Eqs. (2.13) and (2.14), and the question of sufficient hyperbolicity conditions remains open.

3. The Case of a Strong Jump in Density. We shall consider a situation where the density of the fluid in the lower layer ρ_1 is much greater than the density of the fluid in the upper layer ρ_2. In addition, we assume that the line of separation of the fluids h and the pressure at the upper bound of the channel p^* varies smoothly, depending on x. In system (1.3), we stretch $p^* \to \rho_2 p^*$ and multiply the first equation by ρ_1^{-1} and the second equation by ρ_2^{-1}. Here, the small parameter $\alpha = \rho_2\rho_1^{-1}$ arises. Passing formally to the limit as $\alpha \to 0$, we obtain the equations

$$u_T + uu_X + vu_Y + h_X = 0, \qquad h_T + \Big(\int_0^h u\,dY\Big)_X = 0 \tag{3.1}$$

in the lower layer ($0 \leqslant Y \leqslant h$) and the equations

$$u_T + uu_X + vu_Y + p^*_X = 0, \qquad h_T + \Big(\int_1^h u\,dY\Big)_X = 0 \tag{3.2}$$

in the upper layer ($h \leqslant Y \leqslant 1$). System (3.1) describes the plane-parallel motion of a homogeneous fluid with a free boundary in the gravity field in a shallow-wave approximation [5] (u and h are the desired quantities), and Eq. (3.2) describes similar flows in a curved channel [6] (h is a known function, and u and p^* are the desired variables). During modeling, system (1.3) splits into Eqs. (3.1) and (3.2). The line of separation of the fluids h is formed under the influence of only the heavy fluid located in the lower layer, and the upper, quite lightweight fluid moves in a region with specified boundaries. We consider vortex flows with a monotone-in-depth velocity profile in each layer. In contrast to the assumption made in Sec. 2, the intervals of variation in the function u in the lower and upper layers can intersect.

In this case, the characteristic equation (2.9) is simplified and takes the form

$$k\Big(1 - \int_0^{1/2} (u-k)^{-2} u_\lambda \theta\,d\lambda\Big) \int_{1/2}^{1} (u-k)^{-2} u_\lambda \theta\,d\lambda = 0. \tag{3.3}$$

We rewrite Eqs. (3.1) and (3.2) in the Eulerian–Lagrangian coordinates t, x, and λ:

$$u_t + uu_x + \int_0^{1/2} H_x\,d\lambda = 0, \qquad H_t + (uH)_x = 0; \tag{3.1'}$$

$$u_t + uu_x - (u_{1t} + u_1 u_{1x}) = 0, \qquad H_t + (uH)_x = 0, \qquad \int_{1/2}^{1} H\,d\lambda = 1 - h(t,x). \tag{3.2'}$$

We introduce the complex functions $\chi_1(z) = 1 - \int\limits_0^{1/2} (u-z)^{-2} u_\lambda \theta \, d\lambda$ and $\chi_2(z) = \int\limits_{1/2}^{1} (u-z)^{-2} u_\lambda \theta \, d\lambda$ and calculate their limiting values from the upper and lower half-planes on the segments $[u_0, u_{1/2}^-]$ and $[u_{1/2}^+, u_1]$, respectively:

$$\chi_1(u)^{\pm} = 1 + \theta_{1/2}^-(u_{1/2}^- - u)^{-1} - \theta_0(u - u_0)^{-1} - \int\limits_0^{1/2} \theta'_\nu (u' - u)^{-1} \, d\nu \mp \pi i \theta_\lambda / u_\lambda,$$

$$\chi_2(u)^{\pm} = -\theta_1(u_1 - u)^{-1} + \theta_{1/2}^+(u_{1/2}^+ - u)^{-1} + \int\limits_{1/2}^{1} \theta'_\nu (u' - u)^{-1} \, d\nu \pm \pi i \theta_\lambda / u_\lambda.$$

Using these functions, in terms of which Eq. (3.3) can be represented in the form $k\chi_1(k)\chi_2(k) = 0$, we formulate the hyperbolicity conditions for Eqs. (3.1′) and (3.2′).

According to [5], the condition

$$\Delta \arg \chi_1^+(u)/\chi_1^-(u) = 0, \qquad \chi_1^{\pm} \neq 0 \tag{3.4}$$

(the increment of the argument upon variation of u from u_0 to $u_{1/2}^-$) is necessary and sufficient for the hyperbolicity of system (3.1′) if u and θ are differentiable, and u_λ, θ_λ, and H are the Hölder functions with respect to the variable λ.

Equations (3.2′) differ from those considered in [6] by the fact that the specified function h depends not only on x, but also on t. Therefore, in going to the new variables u_λ and θ, one can write system (3.2′) in the form $\boldsymbol{U}_t + A\boldsymbol{U}_x = \boldsymbol{V}$, where $\boldsymbol{U}$ is the vector of the desired quantities, A is the matrix with operator coefficients (the same as in [6]), and $\boldsymbol{V} = (-u_\lambda(1-h)^{-1}h_t, 0)^t$ is the right-hand part. The eigenfunctionals correspond to the data in [6] and can be obtained from the functionals $F_1^{2\lambda+}$, $F_2^{2\lambda+}$, F_1^0, and F_2^0 by means of transformations and a transition to the limit $\alpha \to 0$. Conditions (3.4), in which χ_1 should be substituted instead of the function χ_2, are necessary and sufficient for the hyperbolicity of system (3.2′) with the same requirements for the smoothness of the required functions.

Thus, the equations considered are hyperbolic if definite conditions are realized. Their violation leads to the appearance of complex characteristic roots, which allows one to construct examples of Hadamard's incorrect formulation of the Cauchy problem and, probably, to speak of the occurrence of the shallow-wave instability.

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